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EQUILIBRIUM IN SPATIAL VOTING:

THE MEDIAN VOTER RESULT IS AN ARTIFACT

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The multidimensional spatial theory of electoral competition introduced by Davis and Hinich has received considerable scholarly interest in recent years.¹ The theory, which is a formalized extension of the pioneering efforts of Black [3] and Downs [9], rests on two key assumptions: 1) voters share a common coherent perceptual spatial framework for candidates, and 2) the indifference contours of a voter's utility function are ellipsoids. As a consequence a voter's ideal (bliss) point is an interior point in the space. If all the ellipsoidal indifference contours are similar, these assumptions imply that choice can be rationalized by simple Euclidean distance. In a more general spatial model, weights on the dimensions of the space vary in the population.

The dimensions are described as salient political issues in previous expositions of the theory, but it is more consistent with empirical studies of voter attitudes to conceive of the dimensions as heuristic factors which are used by a voter to forecast a candidate's behavior with respect to economic and social policy once elected to office. It is rational for a voter to simplify the evaluation process by reducing the complexity of the issue space. Since the choice is over

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representatives and not issues per se, a rational voter must forecast how a candidate will behave in office. It is reasonable to use past performance and past associations as a guide to a candidate's future behavior. Moreover most voters do not have much incentive to invest in information, given the small impact of a single vote and the infrequency of elections. Thus a simple rule of thumb based on inexpensive but noisy information is the best evaluation and choice strategy for most voters. For example, the political and social resolution of past civil rights conflicts has involved significant changes in the income and social environment for many nonwhites and certain white groups. As a result of many years of social conflict many white members of organized labor withdrew support from Humphrey in 1968 and especially McGovern in 1972 even though they believed that the Democratic Party supported their economic interests much more than the Republicans.²

Several empirical studies using new spatial mapping techniques on data from the 1968 election survey, give rough support to mapping a position on a specific issue as a combination of economic and social factors, allowing for perceptual uncertainty and error.³

Returning to the theory, the major results deal with a two candidate election where each citizen votes for the candidate who is closest to his ideal point, and the winning candidate is the one who receives the most votes. When the voters have single peaked (quasi-concave) utility functions, then the median voter position is a unique Nash equilibrium for the zero sum political game, assuming an odd number of distinct voters. When the median position is not unique, any median point is an equilibrium. The median voter result and its

social choice extension is the best known result in the theory.⁴ If there is more than one dimension, there is no pure strategy equilibrium unless the voter ideal points are radially symmetric.⁵ For a multi-dimensional space where mixed strategies for candidates are allowed, McKelvey and Ordeshook argue that even if cycles can exist, the candidate positions will be near the point whose j th coordinate is the median of the distribution of the j th coordinates of the ideal points.⁶

These results are essentially deterministic and both voters and candidates have perfect information. Given the perceptual ambiguity of the basic dimensions, it is reasonable to allow some uncertainty in voter choice even if candidate positions are well-defined. If voters choose probabilistically as a function of their distances to the candidates, Hinich, Ledyard and Ordeshook present sufficient conditions for convergence to the mean ideal point by two candidates who seek to maximize their expected plurality.⁷ The mean depends on the positions of extremists much more than the median, unless the distribution is symmetric.

For an asymmetric unidimensional distribution the mean and the median can be nearly one standard deviation apart. It will be shown in this paper that in general, the median is not an equilibrium if there is an additive uncertainty element in the distance from voter to candidate in the one dimensional model. The median becomes the equilibrium when all uncertainty is removed, resulting in a discontinuity in the choice rule. When uncertainty is present in the choice rule, the median is an equilibrium only for a special form of the utility function. Conditions for the mean position to be the political outcome are also

presented. The last section of this paper rationalizes the uncertainty term and voting in terms of private gains from voting for the winner.

1. A Spatial Theory of Voting

Using vector notation for points in R^n , an n dimensional Euclidean space, define the metric

$$\|\theta\|_A = [\theta' A \theta]^{1/2},$$

where A is a symmetric positive definite matrix. For each citizen c , define a generalized difference in distances to the candidates as follows:

$$a_1(c) - a_2(c) + M(\|\psi - x(c)\|_{A(c)}) - M(\|\theta - x(c)\|_{A(c)}) \quad (1)$$

where M is a monotonically increasing function, and the ideal point $x(c)$ and the matrix $A(c)$ can vary among citizens. The parameter $a_i(c) \geq 0$ represents the citizen's belief about the executive ability of the candidate.⁸ When a candidate is virtually unknown, assume that his "a" is sufficiently negative to swamp the spatial term in (1). In order to capture the information effect on a citizen's choice, assume that a_1 increases as the citizen becomes more aware of the i th candidate's record in comparable positions. The voter prefers the candidate he believes will be more effective when elected, other factors being equal.

Suppose that citizens who have identical preferences have different values of $a_1 - a_2$. For each value of x and A , the conditional cumulative distribution function of $\varepsilon = a_1 - a_2$ is denoted by F_ε , i.e.

$$\Pr \{x \text{ votes for } \theta\} = F_\varepsilon [M(\|\psi - x\|_A) - M(\|\theta - x\|_A)] \quad (2)$$

is the conditional probability that a voter whose ideal point is x and whose matrix of dimension weights is A chooses candidate one. The conditional probability that the voter chooses candidate two is just one minus the above probability. When all voters collapse their distribution of ε around a particular value $\tilde{\varepsilon}$, then it follows from (2) that a voter chooses candidate one if

$$M(\|\psi - x\|_A) - M(\|\theta - x\|_A) > \tilde{\varepsilon}$$

and votes for candidate two if the inequality sign is reversed. The choice is indeterminate for equality, so assume for simplicity that the citizen abstains if equality holds. When $\tilde{\varepsilon} = 0$, the choice rule yields the old spatial voting rule for any monotonic M -- vote for the closest candidate in the space (see Figure 1).

When voters choose non-deterministically, the functional form of M matters. The two functions which conform to previous spatial models are $M(y) = y^2$ and $M(y) = |y|$. It is simpler to compare these two models when the space is unidimensional. For $|\psi - x| - |\theta - x| = \delta > 0$, the conditional probability that a voter whose ideal point is x votes for θ

$$\Pr \{x \text{ votes for } \theta\} = F_\varepsilon (\delta^2 + 2\delta|\theta - x|) \quad (3)$$

from (2). This conditional probability is quasi-concave in $|\theta - x|$, i.e. when θ and ψ are a fixed distance apart with respect to a voter's ideal point, the probability that the voter chooses θ increases as θ moves away from the ideal point, but with a diminishing rate. In other words, an extremist voter is more likely than a centrist to choose his closest alternative, given alternatives which are fixed distance apart in the center. The random component has a bigger effect on the choice function of voters who are more satisfied with the candidate positions than voters who are less satisfied; a not unreasonable behavioral property, but not a well established fact for voters.

For the absolute value model, on the other hand,

$$\Pr \{x \text{ votes for } \theta\} = F_{\epsilon}(\delta) \quad (4)$$

for $|\psi - x| - |\theta - x| = \delta$. This probability is obviously independent of θ , which is a very special property of the absolute value model. In addition, the marginal probability $\frac{\partial F_{\epsilon}}{\partial \theta}$ has a discontinuity and sign change at $\theta = x$ for each ψ , provided that $f_{\epsilon}(|\psi - x|) \neq 0$ where f_{ϵ} denotes the conditional density function of ϵ given x and A .

Before comparing these two models in terms of the majority rule policy positions, the objectives of the candidates have to be defined.

2. Candidate Objective Functions

Conceptualize the electorate as a random sample from an infinite population where N , the expected number of voters, is independent of θ and ψ . Letting E denote the expectation with respect to the joint distribution of x and A for the voter

population, the plurality (vote difference) for θ is given by

$$\phi^1(\theta, \psi) = N \{ 2 E F_{\epsilon} \left(M \left(\|\psi - x\|_A \right) - M \left(\|\theta - x\|_A \right) \right) - 1 \}, \quad (5)$$

where f_{ϵ} , the density of ϵ , is assumed to be continuous in order to simplify exposition. Recall that F_{ϵ} is the conditional probability that citizens with identical preference parameters vote for candidate one.

In a two candidate election, the winner is the candidate who receives a majority of the votes cast, i.e. a positive plurality. In the face of uncertainty about voter reactions, there are several reasonable candidate objective functions which are consistent with the definition of winning.⁹ For this paper assume that a candidate attempts to maximize his expected plurality which is equivalent to maximizing expected vote since there are no rational abstainers. The candidates are in a zero sum continuous game which has a pure strategy equilibrium if ϕ^1 is continuous and quasi-concave in θ and quasi-convex in ψ for each (θ, ψ) in a closed convex set in R^n .¹⁰ There are a variety of restrictions on the form of F_{ϵ} and the measure on the voter population which will imply the existence of a pure strategy equilibrium.

For a single issue dimension, it is easy to verify that the Downs result holds for the absolute value model $M(y) = |y|$, i.e. both candidates choose the median ideal point since it is a Nash equilibrium in the zero-sum expected plurality (or equivalently, vote maximizing) political game. When the space has more than one dimension, the deterministic restrictive symmetry conditions for equilibrium at the

mean holds for the absolute value model. It will now be shown for the quadratic model that if an equilibrium exists, it must be at the mean. Consequently the unidimensional median voter result depends on the shape of the probability of voting function when voting is not a discontinuous deterministic function of the distances to the candidates.

Theorem 1. Let $D(x, \theta, \psi) = \|\psi - x\|_A^2 - \|\theta - x\|_A^2$, and suppose that $f_\epsilon(D(x, \theta, \psi)) > 0$ with positive probability for all θ and ψ in R^n . Assume that an equilibrium exists to the zero-sum expected plurality political game. Then the equilibrium is unique and both candidates choose the point

$$\alpha = [E\{f_\epsilon(0)A\}]^{-1} E\{f_\epsilon(0)Ax\}.$$

If $f_\epsilon(0)$ is independent of x and A , as is the case when $f_\epsilon(0)$ is constant in the population, then

$$\alpha = [E(A)]^{-1} E(Ax), \quad (6)$$

Consequently if x and A are uncorrelated, as is trivially the case when A is a constant, then $\alpha = E(x)$, the mean ideal point.

Proof: Suppose that θ^* and ψ^* are an equilibrium strategy pair. Then they must satisfy the first order conditions

$$\frac{\partial}{\partial \theta} \phi^1(\theta^*, \psi^*) = \frac{\partial}{\partial \psi} \phi^2(\theta^*, \psi^*) = 0$$

Thus

$$\begin{aligned} E f_\epsilon(D(x, \theta^*, \psi^*)) A (\theta^* - x) &= 0 \\ E f_\epsilon(D(x, \theta^*, \psi^*)) A (\psi^* - x) &= 0 \end{aligned} \quad (7)$$

Since A is positive definite for each x and $f_\epsilon(D(x, \theta^*, \psi^*)) > 0$, $E f_\epsilon(D(x, \theta^*, \psi^*)) A$ is positive definite and thus is non-singular. It follows from (7) that $\theta^* = \psi^*$, and thus $D(x, \theta^*, \psi^*) = 0$.

Now suppose $f_\epsilon(0)$ is independent of x and A . Then $E f_\epsilon(0)$ is a constant which can be divided out of both sides of the equation, and thus equation (6) holds.

At equilibrium, the expected plurality to one is

$$\phi^1(\alpha, \alpha) = N [2 E F_\epsilon(0) - 1] \quad (8)$$

Suppose that $F_\epsilon(0) > 1/2$ for each x and A , i.e. $a_2 < a_1$ for a majority of voters. Consequently $\phi^1(\alpha, \alpha) > 0$ from (8); a victory for one. The assumption that $F_\epsilon(0) > 1/2$ means that a majority of voters are more certain about candidate one than candidate two. In any case it is in the interest of a candidate to try to increase his "a" level in the population. Since candidates raise a great deal of their campaign contributions by supporting special interests, the quest for contributions will pull the candidates apart if the public goods positions of the candidates are perceived to be effected by their private bargains and the population preferences are asymmetric. Incorporating the effect of contributions into the game is best left for another paper.

Although in general an equilibrium does not exist for the quadratic model, it will be shown that (α, α) is an equilibrium if f_ϵ is a normal density whose mean is zero and whose variance σ^2 is small. If, however, the mean of ϵ is positive, and thus

candidate one has a positive expected plurality, then candidate one can increase his plurality by moving away from α .

Lemma. Suppose that for all x and A , f_ε is a normal density centered at $\varepsilon=0$. If both candidates choose α , then neither candidate has an incentive to make a small move. This local equilibrium is unique.

Proof: The result follows since ϕ^1 and ϕ^2 are locally concave in θ and ϕ respectively at (α, α) .

Let f_ε be normal with mean zero and variance σ^2 , and let $\nabla_{\theta\theta}\phi^1$ denote the matrix $\frac{\partial^2}{\partial\theta_1\partial\theta_j}\phi^1$. Then

$$\nabla_{\theta\theta}\phi^1 = -2NE\{f_\varepsilon\left(\|\psi-x\|_A^2 - \|\theta-x\|_A^2\right)[A + 2\sigma^{-2}\left(\|\psi-x\|_A^2 - \|\theta-x\|_A^2\right)A(\theta-x)(\theta-x)'A]\}$$

Thus when $\theta=\psi$,

$$\nabla_{\theta\theta}\phi^1 = -2NE\{f_\varepsilon(0)A\},$$

which is negative definite. Similar calculations show that $\frac{\partial^2}{\partial\psi_1\partial\psi_j}\phi^2$ is negative definite when $\theta=\psi$. Since $\theta=\psi=\alpha$ uniquely satisfies the first order conditions, the local concavity implies local stability at α .

Now suppose that the mean of ε is $\mu>0$. For simplicity let $n=1$ and $A=1$. Then for $\theta=\psi=\alpha$,

$$\frac{\partial^2}{\partial\theta^2}\phi^1 = -2NE\{f_\varepsilon(0)[1-2\sigma^{-2}\mu(\alpha-x)^2]\}.$$

If $\mu=\sigma$ and σ is small, then

$$\frac{\partial^2}{\partial\theta^2}\phi^1 \approx \frac{4N}{\sqrt{2\pi}}e^{-1}\mu^{-2}\sigma_x^2 > 0 \quad (9)$$

where $\sigma_x^2 = E(\alpha-x)^2$. Thus (9) shows that candidate one will move away from α for this special case.

When the mean of ε is zero, and thus the candidates are symmetric with respect to the non-spatial factors, then the equilibrium at (α, α) is global when σ is small.

Theorem 2. Let $p(x)$ denote the density function of the voter ideal points. Suppose that for all $x \in X$, a closed and bounded subset of R^n , $p(x) > 0$ and $p(x) = 0$ for $x \notin X$. For the quadratic model with a normal ε whose variance is σ^2 , let $\theta^*(\sigma)$ denote the plurality maximizing position for candidate one when two chooses ψ . As $\sigma \rightarrow 0$, $\theta^*(\sigma) \rightarrow \psi$. Thus when σ is small, the mean x is a unique global equilibrium.

Proof: Since this result is so counter intuitive, the proof will be given for the special case when $n=1$ in order to simplify the exposition. The proof for $n \geq 2$ is straightforward using ordinary matrix algebra and some knowledge of the multivariate normal distribution.

Set $\psi = 0$ with no loss of generality. Since $D(x, \theta, 0) = 2\theta x - \theta^2$, θ^* satisfies the first order condition

$$\int_{-\infty}^{\infty} (\theta - x) (\sqrt{2\pi}\sigma)^{-1} \exp\left[-\frac{1}{2\sigma^2}(2\theta x - \theta^2)^2\right] p(x) dx = 0, \quad (10)$$

where $p(x) = 0$ outside X . Since

$$\exp\left[-\frac{1}{2\sigma^2} (2\theta x - \theta^2)^2\right] = \exp\left[-\frac{4\theta^2}{2\sigma^2} \left(x - \frac{\theta}{2}\right)^2\right],$$

it then follows from (10) that if $\theta^* \neq 0$,

$$E_{\theta^*}(\theta^* - x) p(x) = 0 \quad (11)$$

where E_{θ^*} denotes the expectation with respect to a normal density of x whose mean is $\frac{\theta^*}{2}$ and variance is $\frac{\sigma^2}{4\theta^{*2}}$. Rewriting (11),

$$\frac{\theta^*}{2} = \frac{E_{\theta^*}(x - \frac{\theta^*}{2})p(x)}{E_{\theta^*}p(x)},$$

which yields the bound

$$\theta^{*4} \leq \sigma^2 \frac{E_{\theta^*} p^2(x)}{[E_{\theta^*} p(x)]^2} \quad (12)$$

by the Schwarz inequality

$$\begin{aligned} [E_{\theta^*}(x - \frac{\theta^*}{2}) p(x)]^2 &\leq E_{\theta^*}(x - \frac{\theta^*}{2})^2 E_{\theta^*} p^2(x) \\ &= \frac{\sigma^2}{4\theta^{*2}} E_{\theta^*} p^2(x) \end{aligned}$$

Now suppose that $\theta^*(\sigma)$ does not converge to $\psi = 0$ when $\sigma \rightarrow 0$. Then there exists a sequence σ_k such that $\sigma_k \rightarrow 0$ and $\theta^*(\sigma_k) \rightarrow \theta \neq 0$. Thus the variance $\frac{\sigma_k^2}{4(\theta^*(\sigma_k))^2} \rightarrow 0$, which implies that $E_{\theta^*} p(x) \rightarrow p(\frac{\theta}{2})$ and $E_{\theta^*} p^2(x) \rightarrow p^2(\frac{\theta}{2})$, as $\sigma_k \rightarrow 0$. As a

result, the right hand side of (12) converges to zero, yielding a contradiction that $\theta \neq 0$. This argument also shows that $\theta^*(\sigma)$ converges to zero at least as fast as σ .

Now suppose that $\psi = \alpha$. From the above limit result, σ can be made sufficiently small so that θ^* is in the concave region of ϕ^1 about (α, α) , and thus $\phi^1(\alpha, \alpha) > \phi^1(\theta^*, \alpha)$. Since θ^* maximizes $\phi^1(\theta, \alpha)$ by definition, $\theta^* = \alpha$ and thus (α, α) is an equilibrium. The uniqueness follows from the uniqueness of the local stability.

An intuitive explanation is in order for this result. First, it is easy to show why $\theta \rightarrow \psi$ when $\sigma = 0$. If candidate two takes a position to the left of the median, candidate one maximizes his vote (and plurality) by taking a position just to the right of ψ (see Figure 2). When σ is small, the two candidates end up at the mean rather than the median as a result of higher probability of the voters in the tails of $p(x)$ voting for the candidate who is closer to them, as compared with the voters around the median. This difference between centrist and extremist voters is due to the quadratic $(\theta - x)^2$ in the choice function. The absolute value model yields the median as the equilibrium. A small amount of error in the choice rule is sufficient to destroy the generality and elegance of the Black-Downs unidimensional deterministic result. Unless the reader is willing to accept either the quadratic or the absolute value model, it is difficult to say anything about the outcome of majority rule voting using the spatial model with the uncertainty element in it. The quadratic has the advantage over the absolute value model of generalizing the equilibrium to a multidimensional choice setting.

3. Information about the Location of the Mean

It is easy to write down assumptions about the distribution of the ideal points, but it is another matter to estimate this distribution from data which exists, or even hypothetical data which could be collected in principle. The Cahoon, et al. method using candidate "feeling thermometer" scores to develop a spatial map of candidates and voters has been referenced. Thermometer scores are gathered by asking individuals to respond on a 0°-to-100° scale how they feel towards each of a set of candidates. Scores above 50° indicate a "warm" or favorable feeling, while scores less than 50° indicate a "cold" feeling towards a candidate. A score of 50° is supposed to be an indication of being "lukewarm," but the data suggests some confounding with "don't know." The major advantage of thermometer data over more in-depth interviewing is the speed of collection which results in a larger sample size. The noise in the thermometer scores which obscures individual voter preferences is overcome by averaging over a sample which is much larger than that which can be obtained by in-depth interviewing at the same total cost. As long as there exists some common space for the electorate, precision of prediction can be obtained from a large sample of noisy data. Moreover the thermometer question taps the same response mechanism which candidates probe in speeches to a live audience. For example an enthusiastic response after a candidate's speech is related to a high average thermometer score in a survey. It will now be shown that a spatial model of thermometer scores can be used to find the mean ideal position even when candidates do not know the distribution of ideal points for specific issues.

Suppose that the thermometer score for the i th candidate obtained from the k th respondent ($k=1, \dots, N$) is given by

$$\log T_{ki} = \beta_{ki} - \|\theta_i - x_k\|_{A_k}^2 + \varepsilon_{ki}, \quad (13)$$

where ε_{ki} is a zero mean noise term, and β_k is an idiosyncratic parameter of no basic interest. Assume that x_k and A_k are uncorrelated in the population. Thus the equilibrium is $\alpha=0$, where for simplicity the origin of the space has been shifted to make $E(x)=0$. If x_k , A_k , and ε_{ki} have finite variances, then by the central limit theorem the sample mean $\log T_{ki}$ for large N is

$$\overline{\log T_{ki}} = \overline{\beta_{ki}} - \overline{\theta_i' A_k \theta_i} - \overline{x_k' A_k x_k} + O(N^{-\frac{1}{2}}) \quad (14)$$

for each i , where the overbars denote averages over the random sample of N respondents. It then follows from (14) that for large N , the candidate with the highest average log thermometer score is closest to the population mean ideal point. Moreover $\log T_{ki}$ is a linear function of $\|\theta_i\|_{\bar{A}}^2$. Any move a candidate can make which will increase his average log score is a move towards the mean ideal point. A candidate does not have to know where the mean ideal position is on various salient issues in order to move to the overall mean. As long as the aggregate feelings of voters to candidates is related to the spatial model of utility, the mean ideal position can be at least roughly estimated from empirical observations of the relationship between issue stands and high average responses. The quadratic spatial model connects a local plurality equilibrium with popular acclaim.

The next section rationalizes voting in terms of a citizen's utility for political participation.

4. Voting as an Act of Contribution

The act of voting is the least expensive form of political participation for most people. Suppose that a voter contributes resources in order to change the utility he receives if his candidate wins, rather than acting to change the probability of winning, which is taken as given.¹¹ For a large contributor, utility can be in the form of private benefits derived from his association with the candidate. For the small contributor, the utility is a personal satisfaction gained from giving up some resources to provide some support to a candidate. In this paper, voting and giving is connected by the assumption that voting is the lower limit of participation when the cost of voting and the voter budget constraint goes to zero.

Suppose that a voter contributes $r_1 \geq 0$ resource units to candidate one and $r_2 \geq 0$ to candidate two, where the voter's budget constraint is $r \geq r_1 + r_2$. It will now be shown that as the contribution budget goes to zero, the citizen will contribute to at most one candidate, and thus the assumption that the voter can contribute to both candidates is made to allow hedging by the contributor without effecting the voting results when r approaches zero.

Suppose that every potential voter, regardless of personal preferences, perceives the candidate positions as points θ and ψ , respectively, in an n dimensional Euclidean space whose axes are

related to public goods issues. Let $u^1(c, \theta, r_1)$ denote the net utility which citizen c derives from having contributed r_1 to one and one wins.¹² Let $u^2(c, \psi, r_2)$ denote the net utility which the citizen derives from having contributed r_2 to two and two wins. Writing $u^1(c, \theta, r_1)$ as a function of r_1 , but not r_2 implicitly assumes that if the voter contributes resources to both candidates, the contribution to ψ does not effect the utility derived from a contribution to θ . However the limiting result will be the same if a positive r_2 has a negative effect on $u^1(c, \theta, r_1)$. A similar assumption holds for u^2 .

At the beginning of an election campaign there is considerable uncertainty about the preferences of the electorate, as well as their perceptions of the candidates and issues. Formal and informal surveys help the candidates to estimate voter perceptions and preferences, but some uncertainty remains until the election results are in. As the campaign progresses the candidates and voters learn about each other, but the voters are not only uncertain about the outcome, they are uncertain about the willingness and the ability of either candidate to execute his stated or implied policy positions. At campaign's end the winner often can be accurately predicted using paired comparison polls, but the candidate positions are then highly constrained and most voters have made up their minds about the candidates and their chances of election.

Although the final outcome is a function of the policy positions of the candidates and voter preferences according to the rationality paradigm, the dynamics of the electoral process involves

multilayered uncertainty relationships between voter behavior and final output. Suppose that each citizen has a subjective a priori value of p , the probability that candidate one wins. For a given value of p the expected net utility is

$$U(c, \theta, \psi, r_1, r_2) = pu^1(c, \theta, r_1) + qu^2(c, \psi, r_2),$$

where $q = 1 - p$. If u is a strictly concave function of the contribution, then U is a strictly concave function of r_1 and r_2 in a compact convex set and thus has a unique maximum. The main reason for modeling voting in terms of contributions is to introduce the probability of winning into the voting decision.

This model is best rationalized if voting is not secret.

Since most ballots are secret, assume that a voter who is willing to consider the policy positions of both candidates gains in utility by voting for the winner or has a utility loss by voting for the loser. The changes in utility need only be infinitesimal for the result to hold.

Not all citizens, however, are prepared to vote for either candidate depending on their policy positions. Define a partisan of candidate one, a citizen with $\frac{\partial}{\partial r} u^2(c, \psi, 0) < 0$ for all ψ , and $\frac{\partial}{\partial r} u^1(c, \theta, 0) > 0$ for all θ . Thus by the concavity of u^2 in r_2 , $\frac{\partial}{\partial r} u^2(c, \psi, r_2) < 0$ for $r_2 > 0$, and consequently this citizen will not contribute to or vote for candidate two regardless of his policy positions. Since partisans have a fixed voting rule, they do not affect the strategy decisions of the candidates, and consequently can be ignored in the results dealing with equilibria.

Returning to the swing voter, if the expected net utility U is maximized at a point $r_1^* > 0$, $r_2^* > 0$, then the first order conditions can be written

$$\frac{\frac{\partial}{\partial r} u^1(c, \theta, r_1^*)}{\frac{\partial}{\partial r} u^2(c, \psi, r_2^*)} = \frac{q}{p} \quad (15)$$

Theorem 3. As the budget constraint $r \rightarrow 0$, the probability that a citizen gives to and votes for candidate θ if and only if

$$\log \frac{\partial}{\partial r} u^1(c, \theta, 0) - \log \frac{\partial}{\partial r} u^2(c, \psi, 0) > \log \frac{q}{p}$$

where the marginal utilities are positive since the citizen is assumed to be a non-partisan voter.

Proof: For fixed θ and ψ , let $\eta^i(r) = \frac{\partial}{\partial r} u^i$. Suppose that r is sufficiently small so that $r_2^* = r - r_1^*$ (no saturation). Differentiating (15) with respect to p , it follows from the concavity of u^1 in r_1 that $\frac{dr_1^*}{dp} > 0$. When $p = \bar{p}$,

$$\bar{p}(r) = \frac{\eta^2(0)}{\eta^1(r) + \eta^2(0)}, \quad (16)$$

the constraint is reached since $\bar{p}^1(r) = \bar{q}\eta^2(0)$. Thus for all $p \geq \bar{p}$, $r_1^* = r$ and $r_2^* = 0$.

Moreover, as r decreases to zero, $\bar{p}(r)$ decreases since from (16), $\frac{d\bar{p}}{dr} > 0$. In the limit, $\bar{q}(0)/\bar{p}(0)$ is just $\eta^1(0)/\eta^2(0)$ and thus the result follows by taking logarithms.

A citizen will abstain if $\frac{\partial}{\partial r} u^1(c, \theta, 0) < 0$ and $\frac{\partial}{\partial r} u^2(c, \psi, 0) < 0$, i.e. the fixed cost of voting is greater than the expected benefits. In spatial models this form of abstention is called alienation. In sharp contrast to the Hinich, et al model, assume that alienation is independent of the θ and ψ positions. The equilibrium results depend on the a priori distributions of in the population rather than the relationship of turnout to candidate positions.

This voting theory can be related to the new spatial model by assuming that for each c ,

$$\log \frac{\partial}{\partial r} u^1(c, \theta, 0) = \alpha + a_1(c) - \|\theta - x(c)\|_A^2 \quad (17)$$

and

$$\log \frac{\partial}{\partial r} u^2(c, \psi, 0) = \alpha + a_2(c) - \|\psi - x(c)\|_A^2,$$

where α is an arbitrary constant, and $\log \frac{q}{p}$ is added to $a_1 - a_2$ to define the ε in Section 1. As long as voters have different beliefs about the election outcome, $\varepsilon = \log \frac{q}{p} + a_1 - a_2$ will vary in the population even if $a_1 = a_2$ for all citizens. In order, however, for Theorem 2 to still hold, the distribution of p given x and A must be symmetric about $p = 1/2$ since the mean of ε must be zero.

If in addition the utility is separable in θ and r , the utility is then proportional to $\exp \left[-\|\theta - x(c)\|_A^2 \right]$, which is quasi-concave in θ for each r . If $\theta = \psi = x$, the marginal utility for voting for the winner is $\exp(\alpha + a_1)$ if one wins and $\exp(\alpha + a_2)$ if two wins. When both candidates adopt the same policy position, the voter

prefers to support the candidate he anticipates will be more able in the job. The utility function is positive as a result of the separability and non-partisan assumptions.

In contrast to the old spatial model, $A(c)$ is not defined up to a scalar multiple, but has elements whose units are defined by the units of the dimensions. The problem of estimating the parameters of such a model when the units are unknown is discussed by Cahoon, et al., although there is no method developed as yet to relate even the mean ideal point to specific policy positions.

4. Conclusion

When the space is unidimensional, the median is a majority rule equilibrium when the probability of voting depends on the absolute value of the distance to the candidate from each voter. For the quadratic spatial model for voting, on the other hand, it has been shown that the mean ideal point is the only possibility for an equilibrium. When the error term in the spatial model is normal with mean zero, the mean is a local equilibrium, which becomes a global equilibrium when the variance is small. When the mean of the normal error is not zero, however, there is an asymmetry in the voter population which can upset the equilibrium at the mean.

It is not surprising that candidates try to adopt centrist positions when the rule for winning is plurality. When candidate one gains a voter, his plurality over two increases by two votes, resulting in an incentive for the candidates to compete for voters whose ideal points are between the candidate positions (provided there are a

significant number of voters near the center). Majority rule produces outcomes near the mean as long as the bulk of the population is centrist.

When conditions in the model support a global equilibrium at the mean, two candidate competition results in a predictable single outcome for public goods. On the other hand, the theory says nothing about the income redistribution which results from the private bargains made by the winning candidate to individuals and groups on the road to election. A crucial question for scholars of democratic choice is whether the voting process is important when the institution Government becomes such a major factor in our lives.

FOOTNOTES

1. In spatial theory, the citizen's utility for a candidate at θ is a decreasing function of $\|\theta - x(c)\|$, the Euclidean distance between θ and the voter's ideal point (Davis and Hinich [7]. For a review of spatial theory, see Davis, Hinich, and Ordeshook [8].
2. See the analysis of the 1968 election survey by Converse, et al [5]. The 1972 election survey is discussed by Miller, et al [15].
3. The connection between issues and dimensions is discussed in the spatial analysis of the candidate feeling thermometer scores from the 1968 election survey by Cahoon, Hinich, and Ordeshook [4]. Also see Rabinowitz [16].
4. See Arrow's [2] elegant reformulation of single peaked preferences and majority rule social choice of Black [3].
5. Necessary and sufficient conditions are given by Davis, DeGroot, and Hinich [6].
6. See McKelvey and Ordeshook [13].
7. See Hinich, et al [12]. McKelvey [14] generalizes these results and also most of the Davis-Hinich spatial results.

8. Since the electorate is a random sample from an infinitive population with a given probability measure, the "parameters" a_1 , a_2 , x , and A are random variables whose joint distribution is of primary interest even though it depends on the underlying measure.
9. A comparison of equilibria under different candidate objective functions, including expected plurality, is given by Aranson, Hinich, and Ordeshook [1].
10. See Friedman [11].
11. See Riker and Ordeshook [17], and Ferejohn and Fiorina [10].
12. The net utility for citizen c is the utility of the resources derived as a result of candidate θ winning and recognizing the r_1 contribution, minus the total contribution $r_1 + r_2$, which will be exactly r if there is no saturation. Since r will go to zero, the assumption of no saturation is easily met. Moreover in the limit the net is irrelevant, and the slope of the utility function which appears in the model becomes the marginal utility of voting for θ given he wins.

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PROBABILITY OF VOTING FOR ONE

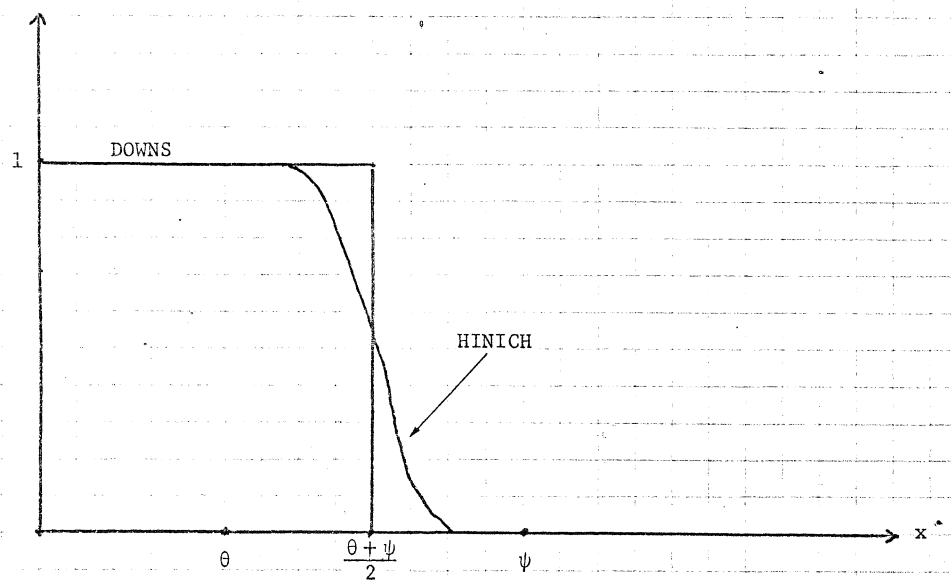


Figure 1

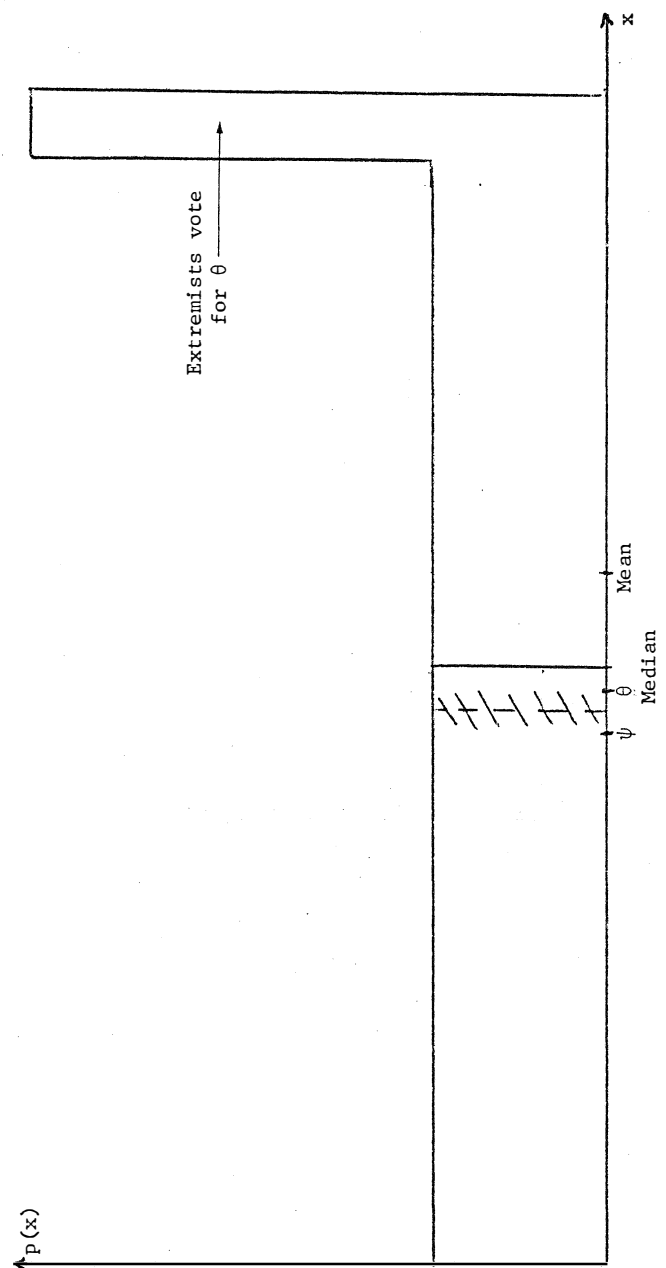


Figure 2